Radiative Tail for Inelastic Electron Scattering*

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The cross-section differential in the energies and angles of all final particles for the bremsstrahlung which accompanies nuclear excitation is derived, in first Born approximation and neglecting nuclear recoil, in terms of the form factors corresponding to the Coulomb and current interactions, which are left arbitrary throughout the calculation. This cross section is integrated over all photon angles, without approximation, and its limit for large energies ($E \gg mc^2$) of the initial and final electron is discussed. As in the case of elastic electron scattering considered previously, photons emitted in the direction of either the incoming or outgoing electron contribute to this integrated cross section terms of order $\ln(E/mc^2)$, while all other photon directions contribute terms of relative order 1. We give explicit expressions for these terms, as well as a numerical evaluation for some typical cases; the formulas are also valid for electron scattering angles ϑ equal or very close to 180°. It is shown explicitely that the logarithmically divergent terms which appear in the integral of our high-energy cross section over the energy of the final electron are cancelled exactly if to this integral one adds the radiative corrections to the inelastic electron scattering cross section.

I. INTRODUCTION

THE Bethe-Heitler cross section with arbitrary form factor has been integrated in a previous paper[1] over all photon angles and without making any approximation, for the case of elastic electron scattering. This result was used to calculate the radiative tail, which is the extension of the elastic peak (or an inelastic peak) in the spectrum of scattered electrons due to the emission of real, hard, photons. We showed how the contribution to the cross section from photons emitted in the direction of either the incoming or the outgoing electron, denoted as "peak contribution," gives terms of order $\ln\epsilon$ (where ϵ is the electron energy expressed in units of mc^2) and that the contribution from all other photon directions, the "background contribution," is of relative order 1.

Since our Paper (I) appeared, Ginsberg and Pratt[2] have performed a calculation similar to ours, but in which they take into account the current interaction as well as the Coulomb interaction for the elastic scattering of electrons, while we dealt only with the Coulomb interaction. Their calculation is not concerned with the separation into peak and background contributions, and therefore the numerical integration for several specific form factors is performed at an earlier stage than in ours, without discussing the relative magnitude of the various terms.

In this paper we extend those two calculations to include the case of inelastic electron scattering, in which the electron loses energy not only by radiation (bremsstrahlung) but also by exciting the nucleus. We will first derive the cross section differential in the energies and angles of all final particles for bremsstrahlung associated with nuclear excitation, and this will be done in first Born approximation and neglecting nuclear recoil. This cross section is given in terms of the form factors corresponding to the nuclear transition, which are left arbitrary throughout the calculation. As we will see, these form factors may be expressed in terms of the reduced matrix element for the given nuclear transition. They take into account both Coulomb and current interactions. As in I, we are concerned here with the radiative tail (emission of real, hard, photons) and not with radiative corrections (emission and reabsorption of virtual photons and emission of real, soft photons). We next perform the integration of the cross section over all photon angles, keeping arbitrary form factors and without approximation. After this we consider in detail the limit of the integrated cross section for large energies of the initial and final electrons, $\epsilon_1 \gg 1$, $\epsilon_2 \gg 1$, the largest neglected terms being of order $1/\epsilon_2^2$. It is shown that, as in the elastic case described in I, there is here a separation of the cross section into peak and background terms. Explicit expressions as well as numerical examples are given for both types of terms. We will also discuss in detail the case of scattering angle very close (or equal) to 180°. In the course of the discussion, the high-energy cross section is integrated over the energy of the final electron, giving logarithmically divergent terms with the factor $\ln(1/\Delta k)$, where Δk is the minimum photon energy included in this integration, in units of mc^2. It is shown explicitly that if to this integral of the radiative tail one adds the radiative corrections to the inelastic electron scattering cross section, then these divergent terms are cancelled exactly.

II. INELASTIC BREMSSTRAHLUNG CROSS SECTION

The cross section for scattering of an electron with emission of a hard photon and excitation of the scattering nucleus is derived here in a manner closely paralleling that of Alder *et al.*[3] who treat the case of electron

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[1] L. C. Maximon and D. B. Isabelle, Phys. Rev. **133**, B1344 (1964), hereafter referred to as I.
[2] E. S. Ginsberg and R. H. Pratt, Phys. Rev. **134**, B773 (1964).
[3] K. Alder, A. Bohr, T. Huus, B. Mottelson, and A. Winther, Rev. Mod. Phys. **28**, 432 (1956). Note in particular Sec. II E.3, pp. 475, 476, in this reference. For further details and a specific application of the analysis of Alder *et al.*, see also Ref. 15.

scattering with nuclear excitation but without photon emission. For the transition matrix element corresponding to their Eq. (II E.33) but including emission of a photon we find

$$H'=\langle f|\mathcal{H}^{(1)}|i\rangle=e\hbar c(4\pi)^{3/2}\left(\frac{Ze^2}{mc^2}\right)\left(\frac{\hbar}{mc}\right)^2\left(\frac{2\pi}{kmc^2}\right)^{1/2}\Bigg\{\sum_{\lambda=1}^{\infty}\sum_{\mu=-\lambda}^{\lambda}\frac{i^{\lambda+1}(-1)^\mu}{\lambda(q^2-\omega^2)}$$
$$\times[\langle u_2|\mathbf{A}_T|u_1\rangle\cdot\hat{q}\times\mathbf{L}_qY_{\lambda\mu}(\hat{q})\langle I_fM_f|\mathfrak{M}(E\lambda,-\mu,q)|I_iM_i\rangle$$
$$-\langle u_2|\mathbf{A}_T|u_1\rangle\cdot\mathbf{L}_qY_{\lambda\mu}(\hat{q})\langle I_fM_f|\mathfrak{M}(M\lambda,-\mu,q)I_iM_i\rangle]$$
$$+\sum_{\lambda=0}^{\infty}\sum_{\mu=-\lambda}^{\lambda}\frac{i^{\lambda+1}(-1)^\mu}{q^2}i\langle u_2|\Lambda_L|u_1\rangle Y_{\lambda\mu}(\hat{q})\langle I_fM_f|\mathfrak{M}(C\lambda,-\mu,q)|I_iM_i\rangle\Bigg\}, \quad (1)$$

where

$$\begin{aligned}\mathbf{A}_T&\equiv(\boldsymbol{\alpha}\cdot\mathbf{e}^*K_2\boldsymbol{\alpha}/d_2)-(\boldsymbol{\alpha}K_1\boldsymbol{\alpha}\cdot\mathbf{e}^*/d_1),\\ \Lambda_L&\equiv(\boldsymbol{\alpha}\cdot\mathbf{e}^*K_2/d_2)-(K_1\boldsymbol{\alpha}\cdot\mathbf{e}^*/d_1),\\ K_1&=\epsilon_1-k+\boldsymbol{\alpha}\cdot(\mathbf{p}_1-\mathbf{k})+\beta,\\ K_2&=\epsilon_2+k+\boldsymbol{\alpha}\cdot(\mathbf{p}_2+\mathbf{k})+\beta,\\ d_1&=2k(\epsilon_1-p_1\cos\theta_1),\\ d_2&=2k(\epsilon_2-p_2\cos\theta_2).\end{aligned} \quad (2)$$

Here, u_1 and u_2 are the normalized plane-wave spinors of the initial and final electron, respectively, $\boldsymbol{\alpha}$ and β are the usual Dirac matrices, and $\mathbf{e}$ is the polarization vector of the emitted photon. Finally,

$$\mathbf{L}_q=-i\mathbf{q}\times\boldsymbol{\nabla}_q \quad (3)$$

and operates on $\mathbf{q}$. The notation used is the following:

$(\epsilon_1,\mathbf{p}_1)$, $(\epsilon_2,\mathbf{p}_2)$: energy and momentum of the initial and final electron, respectively.
k, $\mathbf{k}$: energy and momentum of the emitted photon.
ω, $\mathbf{q}$: energy and momentum transfer to the nucleus.
(θ_1,φ_1), (θ_2,φ_2): polar and azimuthal angles of the initial and final electron, respectively, in a coordinate system with z axis along the direction of the photon, $\mathbf{k}$.
(ϑ_k,ϕ_k), (ϑ,ϕ): polar and azimuthal angles of the photon and final electron, respectively, in a coordinate system with z axis along the direction of the initial electron, $\mathbf{p}_1$.

These angles are related by

$$\begin{aligned}\theta_1&=\vartheta_k,\\ \cos\theta_2&=\cos\vartheta\cos\vartheta_k+\sin\vartheta\sin\vartheta_k\cos(\phi-\phi_k),\\ \cos\vartheta&=\cos\theta_1\cos\theta_2+\sin\theta_1\sin\theta_2\cos(\varphi_1-\varphi_2),\end{aligned} \quad (4)$$

and

$$\omega=\epsilon_1-\epsilon_2-k,\quad \mathbf{q}=\mathbf{p}_1-\mathbf{p}_2-\mathbf{k}. \quad (5)$$

The units of energy and momentum are mc^2 and mc throughout.

The operators for the Coulomb and electric and magnetic current multipole transitions, which have been indicated by $\mathfrak{M}$ in Eq. (1), are expressed in terms of the nuclear charge and current density operators ρ_n and $\mathbf{j}_n$ by

$$\mathfrak{M}(C\lambda,\mu,q)=\frac{(4\pi)^{1/2}}{Ze}\int\rho_nj_\lambda(qr)Y_{\lambda\mu}(\theta,\varphi)d^3\mathbf{r},$$
$$\mathfrak{M}(E\lambda,\mu,q)=\frac{(4\pi)^{1/2}}{Zeq(\lambda+1)}\times\int\mathbf{j}_n\cdot\boldsymbol{\nabla}\times\mathbf{L}_r(j_\lambda(qr)Y_{\lambda\mu}(\theta,\varphi))d^3\mathbf{r}, \quad (6)$$
$$\mathfrak{M}(M\lambda,\mu,q)=-\frac{i(4\pi)^{1/2}}{Ze(\lambda+1)}\times\int\mathbf{j}_n\cdot\mathbf{L}_r(j_\lambda(qr)Y_{\lambda\mu}(\theta,\varphi))d^3\mathbf{r},$$

where

$$\mathbf{L}_r=-i\mathbf{r}\times\boldsymbol{\nabla}_r. \quad (7)$$

These operators are thus related to those of Ref. 3 by a factor[4]:

$$\mathfrak{M}=\frac{q^\lambda}{(2\lambda+1)!!}\frac{(4\pi)^{1/2}}{Ze}\mathfrak{M}_{\text{Alder } et\ al.}. \quad (8)$$

The cross section is then given by

$$d\sigma=\frac{2\pi}{\hbar}\sum_\alpha|H'|^2\frac{\rho_f}{v_i}, \quad (9)$$

where $\rho_f=p_2\epsilon_2d\Omega_2k^2dkd\Omega_k(mc^2)^5/(2\pi\hbar c)^6$ is the density of final states ($d\Omega_2=\sin\vartheta d\vartheta d\phi$ and $d\Omega_k=\sin\vartheta_kd\vartheta_kd\phi_k$ refer to the angles of $\mathbf{p}_2$ and $\mathbf{k}$, respectively) and $v_i=p_1c/\epsilon_1$ is the velocity of the incident electron. Here $\sum_\alpha$ indicates the average over the initial and sum over the final states which are not observed. We shall average over the initial nuclear spin orientation M_i and initial electron spin direction $\boldsymbol{\zeta}_1$, and sum over the final nuclear spin orientation M_f, the final electron spin direction $\boldsymbol{\zeta}_2$, and the photon polarization $\mathbf{e}$. To this

[4] Reference 3, p. 446 Eqs. (II B.16) and (II B.17) and p. 476 Eq. (II E.38).

end we write, following Alder *et al.*,[3] the nuclear transition matrix elements in terms of the reduced matrix elements. We have, for the case of the Coulomb interaction,

$$\langle I_fM_f|\mathfrak{M}(C\lambda,\mu,q)|I_iM_i\rangle = (-1)^{I_f-M_f}\begin{pmatrix} I_f & \lambda & I_i \\ -M_f & \mu & M_i\end{pmatrix}\langle I_f\|\mathfrak{M}(C\lambda,q)\|I_i\rangle \quad (10)$$

with identical equations for the electric and magnetic current transition matrix elements. We insert (10) in (1) and use the orthogonality relations for the $3-j$ coefficients[5]:

$$\sum_{M_i,M_f}\begin{pmatrix} I_f & \lambda & I_i \\ -M_f & \mu & M_i\end{pmatrix}\begin{pmatrix} I_f & \lambda' & I_i \\ -M_f & \mu' & M_i\end{pmatrix} = (2\lambda+1)^{-1}\delta_{\lambda\lambda'}\delta_{\mu\mu'}. \quad (11)$$

The cross section, summed over all final spins and polarizations and averaged over initial spins then becomes

$$d\sigma=\frac{1}{2}\frac{1}{2I_i+1}\frac{2\pi}{\hbar}\sum_{M_i,M_f,\zeta_1,\zeta_2,\mathbf{e}}|H'|^2\frac{\rho_f}{v_i}$$
$$=\frac{1}{(2\pi)^2}\frac{e^2}{\hbar c}\left(\frac{Ze^2}{mc^2}\right)^2\frac{p_2}{p_1}\frac{dk}{k}d\Omega_2 d\Omega_k\left(\frac{16\pi^2 k\epsilon_1\epsilon_2}{2I_i+1}\right)$$
$$\times\tfrac{1}{2}\sum_{\zeta_1,\zeta_2,\mathbf{e}}\Bigg\{\sum_{\lambda=1}^{\infty}\sum_{\mu=-\lambda}^{\lambda}\frac{1}{(2\lambda+1)\lambda^2(q^2-\omega^2)^2}\big[|\langle u_2|\mathbf{A}_T|u_1\rangle\cdot\hat{q}\times\mathbf{L}_qY_{\lambda\mu}(\hat{q})|^2|\langle I_f\|\mathfrak{M}(E\lambda,q)\|I_i\rangle|^2$$
$$+|\langle u_2|\mathbf{A}_T|u_1\rangle\cdot\mathbf{L}_qY_{\lambda\mu}(\hat{q})|^2|\langle I_f\|\mathfrak{M}(M\lambda,q)\|I_i\rangle|^2\big]$$
$$+\sum_{\lambda=0}^{\infty}\sum_{\mu=-\lambda}^{\lambda}\frac{1}{(2\lambda+1)q^4}|\langle u_2|A_L|u_1\rangle Y_{\lambda\mu}(\hat{q})|^2|\langle I_f\|\mathfrak{M}(C\lambda,q)\|I_i\rangle|^2\Bigg\}. \quad (12)$$

In order to perform the sums over μ in (12) we require the following formulas, which are discussed and justified below:

$$\sum_m Y_{lm}^*(\hat{r})Y_{lm}(\hat{r})=(2l+1)/4\pi, \quad (13a)$$

$$\sum_m(\mathbf{A}\cdot\mathbf{L}Y_{lm}(\hat{r}))^*Y_{lm}(\hat{r})=0, \quad (13b)$$

$$\sum_m(\mathbf{A}\cdot\mathbf{L}Y_{lm}(\hat{r}))^*(\mathbf{B}\cdot\mathbf{L}Y_{lm}(\hat{r})) =[l(l+1)(2l+1)/8\pi](\mathbf{A}\times\hat{r})\cdot(\mathbf{B}\times\hat{r}), \quad (13c)$$

$$\sum_m(\mathbf{L}Y_{lm}(\hat{r}))^*\cdot(\mathbf{L}Y_{lm}(\hat{r}))=l(l+1)(2l+1)/4\pi, \quad (13d)$$

$$\sum_m(\hat{r}\times\mathbf{L}Y_{lm}(\hat{r}))^*\cdot(\mathbf{L}Y_{lm}(\hat{r}))=0, \quad (13e)$$

$$\sum_m(\hat{r}\times\mathbf{L}Y_{lm}(\hat{r}))^*\cdot(\hat{r}\times\mathbf{L}Y_{lm}(\hat{r}))$$
$$=\sum_m\{(\mathbf{L}Y_{lm}(\hat{r}))^*\cdot(\mathbf{L}Y_{lm}(\hat{r})$$
$$-(\hat{r}\cdot\mathbf{L}Y_{lm}(\hat{r}))^*(\hat{r}\cdot\mathbf{L}Y_{lm}(\hat{r}))\}$$
$$=l(l+1)(2l+1)/4\pi. \quad (13f)$$

Here $\mathbf{A}$ and $\mathbf{B}$ are arbitrary vectors and $\mathbf{L}=\mathbf{L}_r=-i\mathbf{r}\times\boldsymbol{\nabla}_r$ as in (7). These formulas may all be obtained from[6]

$$\sum_m Y_{lm}^*(\hat{r}')Y_{lm}(\hat{r}'')=[(2l+1)/4\pi]P_l(\cos\chi), \quad (13g)$$

where

$$\mathbf{r}_1'=(r',\theta',\varphi'), \quad \mathbf{r}''=(r'',\theta'',\varphi''),$$

and

$$\cos\chi=\cos\theta'\cos\theta''+\sin\theta'\sin\theta''\cos(\varphi'-\varphi'').$$

Setting $\mathbf{r}'=\mathbf{r}''=\mathbf{r}$ in (13g) gives (13a) at once. Noting that in spherical coordinates

$$\mathbf{L}_r=-i\left(-\mathbf{u}_\theta\frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}+\mathbf{u}_\varphi\frac{\partial}{\partial\theta}\right),$$

operating on both sides of (13g) with $(\mathbf{A}\cdot\mathbf{L}_{r'})^*$ and then setting $\mathbf{r}'=\mathbf{r}''=\mathbf{r}$ gives (13b). Operating on both sides of (13g) with $(\mathbf{A}\cdot\mathbf{L}_{r'})^*(\mathbf{B}\cdot\mathbf{L}_{r''})$ and then setting $\mathbf{r}'=\mathbf{r}''=\mathbf{r}$ gives (13c). The expression in (13d) may be written in the form

$$\sum_m(\mathbf{L}Y_{lm}(\hat{r}))^*\cdot(\mathbf{L}Y_{lm}(\hat{r})) =\sum_j\sum_m(\mathbf{e}_j\cdot\mathbf{L}Y_{lm}(\hat{r}))^*(\mathbf{e}_j\cdot\mathbf{L}Y_{lm}(\hat{r})),$$

where $\mathbf{e}_j$ ($j=1, 2, 3$) forms a set of orthonormal vectors, and hence is of the same form as (13c). To evaluate (13e) we choose $\mathbf{e}_1$ and $\mathbf{e}_2$ to be two mutually

[5] A. R. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton University Press, Princeton, New Jersey, 1957), p. 47, Eq. (3.7.8).

[6] Reference 5, p. 63, Eq. (4.6.6).

orthogonal vectors such that $\mathbf{e}_1\times\mathbf{e}_2=\hat{r}$. Then

$$\begin{aligned}
&\sum_m(\hat{r}\times\mathbf{L}Y_{lm}(\hat{r}))^*\cdot(\mathbf{L}Y_{lm}(\hat{r}))\\
&=\sum_m[(\mathbf{e}_1\times\mathbf{e}_2)\times\mathbf{L}Y_{lm}(\hat{r})]^*\cdot(\mathbf{L}Y_{lm}(\hat{r}))\\
&=\sum_m[\mathbf{e}_2(\mathbf{e}_1\cdot\mathbf{L}Y_{lm}(\hat{r}))-\mathbf{e}_1(\mathbf{e}_2\cdot\mathbf{L}Y_{lm}(\hat{r}))]\cdot(\mathbf{L}Y_{lm}(\hat{r}))\\
&=\sum_m\{(\mathbf{e}_1\cdot\mathbf{L}Y_{lm}(\hat{r}))^*(\mathbf{e}_2\cdot\mathbf{L}Y_{lm}(\hat{r}))\\
&\qquad-(\mathbf{e}_2\cdot\mathbf{L}Y_{lm}(\hat{r}))^*(\mathbf{e}_1\cdot\mathbf{L}Y_{lm}(\hat{r}))\}
\end{aligned}$$

so that we again have sums of the same form as (13c). Finally, (13f) follows from (13c) and (13d).

We then carry out in closed form the sums over ζ_1, ζ_2, $\mathbf{e}$, and μ indicated in (12). The entire dependence of these sums on λ is contained in the simple factor shown explicitly, viz., $2\lambda+1$ in (14) and $\lambda(\lambda+1)(2\lambda+1)$ in (15):

$$\sum_{\mu=-\lambda}^{\lambda}\tfrac{1}{2}\sum_{\zeta_1,\zeta_2,\mathbf{e}}|\langle u_2|\Lambda_L|u_1\rangle Y_{\lambda\mu}(\hat{q})|^2=\frac{1}{4k^2\epsilon_1\epsilon_2}\frac{2\lambda+1}{4\pi}\sigma_L,\quad(14)$$

$$\begin{aligned}
&\sum_{\mu=-\lambda}^{\lambda}\tfrac{1}{2}\sum_{\zeta_1,\zeta_2,\mathbf{e}}|\langle u_2|\mathbf{\Lambda}_T|u_1\rangle\cdot\hat{q}\times\mathbf{L}_qY_{\lambda\mu}(\hat{q})|^2\\
&=\sum_{\mu=-\lambda}^{\lambda}\tfrac{1}{2}\sum_{\zeta_1,\zeta_2,\mathbf{e}}|\langle u_2|\mathbf{\Lambda}_T|u_1\rangle\cdot\mathbf{L}_qY_{\lambda\mu}(\hat{q})|^2\\
&=\frac{1}{4k^2\epsilon_1\epsilon_2}\frac{\lambda(\lambda+1)(2\lambda+1)}{8\pi}\sigma_T.\quad(15)
\end{aligned}$$

The factors σ_L and σ_T, associated with the interaction of the electron with the electromagnetic field, are given by

$$\begin{aligned}
\sigma_L=&\frac{p_1^2\sin^2\theta_1[4\epsilon_2(\epsilon_2+\omega)+\omega^2-q^2]}{(\epsilon_1-p_1\cos\theta_1)^2}+\frac{p_2^2\sin^2\theta_2[4\epsilon_1(\epsilon_1-\omega)+\omega^2-q^2]}{(\epsilon_2-p_2\cos\theta_2)^2}\\
&-\frac{2p_1p_2\sin\theta_1\sin\theta_2\cos(\varphi_1-\varphi_2)[2\epsilon_1(\epsilon_1-\omega)+2\epsilon_2(\epsilon_2+\omega)+\omega^2-q^2]}{(\epsilon_1-p_1\cos\theta_1)(\epsilon_2-p_2\cos\theta_2)}+\frac{2k^2(p_1^2\sin^2\theta_1+p_2^2\sin^2\theta_2)}{(\epsilon_1-p_1\cos\theta_1)(\epsilon_2-p_2\cos\theta_2)},\quad(16)
\end{aligned}$$

and

$$\begin{aligned}
\sigma_T=&\frac{p_1^2\sin^2\theta_1[4\epsilon_2(\epsilon_2+\omega)+q^2-\omega^2-4]}{(\epsilon_1-p_1\cos\theta_1)^2}+\frac{p_2^2\sin^2\theta_2[4\epsilon_1(\epsilon_1-\omega)+q^2-\omega^2-4]}{(\epsilon_2-p_2\cos\theta_2)^2}\\
&-\frac{2p_1p_2\sin\theta_1\sin\theta_2\cos(\varphi_1-\varphi_2)[2\epsilon_1(\epsilon_1-\omega)+2\epsilon_2(\epsilon_2+\omega)+q^2-\omega^2-4]}{(\epsilon_1-p_1\cos\theta_1)(\epsilon_2-p_2\cos\theta_2)}+\frac{2k^2(p_1^2\sin^2\theta_1+p_2^2\sin^2\theta_2)}{(\epsilon_1-p_1\cos\theta_1)(\epsilon_2-p_2\cos\theta_2)}\\
&+4k^2\left[\frac{\epsilon_2-p_2\cos\theta_2}{\epsilon_1-p_1\cos\theta_1}+\frac{\epsilon_1-p_1\cos\theta_1}{\epsilon_2-p_2\cos\theta_2}\right]-\frac{\omega^2}{q^2}\sigma_L.\quad(17)
\end{aligned}$$

From (12) and (15) we then have

$$d\sigma=\frac{1}{(2\pi)^2}\frac{e^2}{\hbar c}\left(\frac{Ze^2}{mc^2}\right)^2\frac{p_2}{p_1}\frac{dk}{k}d\Omega_2d\Omega_k\times\left\{\frac{\mathfrak{F}_L^2(q)}{q^4}\sigma_L+\frac{\mathfrak{F}_T^2(q)}{(q^2-\omega^2)^2}\sigma_T\right\}.\quad(18)$$

The form factors $\mathfrak{F}_L(q)$ (associated with Coulomb interactions with the nucleus), and $\mathfrak{F}_T(q)$ (associated with current interactions), are given in terms of the reduced nuclear matrix elements by

$$\begin{aligned}
\mathfrak{F}_L^2(q)&=\frac{1}{2I_i+1}\sum_{\lambda=0}^{\infty}|\langle I_f\|\mathfrak{M}(C\lambda,q)\|I_i\rangle|^2,\\
\mathfrak{F}_T^2(q)&=\frac{1}{2I_i+1}\tfrac{1}{2}\sum_{\lambda=1}^{\infty}\left(\frac{\lambda+1}{\lambda}\right)[|\langle I_f\|\mathfrak{M}(E\lambda,q)\|I_i\rangle|^2\\
&\qquad+|\langle I_f\|\mathfrak{M}(M\lambda,q)\|I_i\rangle|^2].
\end{aligned}\quad(19)$$

We note at this point that if we set $k=\epsilon_1-\epsilon_2$ $(\omega=0)$ in our Eqs. (16) and (17), we obtain the expressions T_{ch} and T_{mag} given by Ginsberg and Pratt[2] for the elastic case:

$$\sigma_L(\omega=0)=T_{\mathrm{ch}},\quad\sigma_T(\omega=0)=T_{\mathrm{mag}}.$$

Furthermore, if we consider only the Coulomb scattering $(\mathfrak{F}_T=0)$ we find the expression for the inelastic bremsstrahlung cross section obtained by Perez y Jorba.[7]

The general formula (18) has also been obtained by Nguyen Ngoc and Perez y Jorba.[8] However, the technique used in their derivation differs considerably from that used here and in particular is not such as to make explicit the connection between the form factors and the reduced matrix elements. This will be of particular importance if one wishes to consider inelastic

[7] J. P. Perez y Jorba, J. Phys. Rad. **22**, 733 (1961).

[8] H. Nguyen Ngoc and J. P. Perez y Jorba, Phys. Rev. (to be published).

electron scattering in which either the electron or the target nucleus is polarized.

III. INTEGRATION OF THE CROSS SECTION OVER PHOTON DIRECTIONS

Since in an ordinary electron scattering experiment the emitted photon is not observed, we wish to integrate the differential cross section given by (18) over the angles of the photon direction, ϑ_k and ϕ_k. Owing to the presence of the form factors $\mathfrak{F}_L(q)$ and $\mathfrak{F}_T(q)$, the convenient variables for this integration are clearly ϑ_k and q^2. Using (4) and (5) we express the cross section in terms of ϑ_k, q^2, ϑ and ϕ. (In addition, we must multiply the cross section by two, since the entire range of q is covered by letting ϕ_k go from 0 to π.) As in I, the integration over ϑ_k is straightforward. The integration over q^2 is not carried out explicitly at this point, so that we obtain now the cross section for the inelastic scattering of an electron through an angle ϑ:

$$d\sigma=\frac{1}{2\pi}\frac{e^2}{\hbar c}\left(\frac{Ze^2}{mc^2}\right)^2\frac{p_2}{p_1}\frac{dk}{k}\sin\vartheta d\vartheta d\phi \times\int_{q_m^2}^{q_M^2}\left\{\frac{\mathfrak{F}_L^2(q)}{q^4}I_L(q^2)+\frac{\mathfrak{F}_T^2(q)}{(q^2-\omega^2)^2}I_T(q^2)\right\}d(q^2), \quad (20)$$

with

$$\begin{aligned}
I_L(q^2)=&\frac{1}{\pi}\int_0^\pi \sin\vartheta_k d\vartheta_k \sigma_L(\vartheta_k,q^2,\vartheta,\phi)\Big/\left(\frac{\partial(q^2)}{\partial\phi_k}\right)\\
=&-\frac{2k}{|\mathbf{p}_1-\mathbf{p}_2|}-\frac{k}{(\eta+\omega^2-q^2)D_1^{1/2}}\{(\eta+\omega^2-q^2)^2+4(\eta+\omega^2-q^2)[\epsilon_1^2+\epsilon_2(\epsilon_2+\omega)]\\
&\qquad+8k^2-2(\eta+2)[2\epsilon_1(\epsilon_1-\omega)+2\epsilon_2(\epsilon_2+\omega)+\omega^2-q^2]\}\\
&+\frac{k}{(\eta+\omega^2-q^2)D_2^{1/2}}\{(\eta+\omega^2-q^2)^2+4(\eta+\omega^2-q^2)[\epsilon_2^2+\epsilon_1(\epsilon_1-\omega)]\\
&\qquad+8k^2-2(\eta+2)[2\epsilon_1(\epsilon_1-\omega)+2\epsilon_2(\epsilon_2+\omega)+\omega^2-q^2]\}\\
&-\frac{k[4\epsilon_2(\epsilon_2+\omega)+\omega^2-q^2]}{D_1^{3/2}}[(q^2-q_2^2)[\eta+2\epsilon_1(\epsilon_1-\epsilon_2)]+4k\epsilon_1p_2^2\sin^2\vartheta]\\
&\qquad+\frac{k[4\epsilon_1(\epsilon_1-\omega)+\omega^2-q^2]}{D_2^{3/2}}[(q^2-q_1^2)[\eta+2\epsilon_2(\epsilon_2-\epsilon_1)]-4k\epsilon_2p_1^2\sin^2\vartheta], \quad (21)
\end{aligned}$$

$$\begin{aligned}
I_T(q^2)=&\frac{1}{\pi}\int_0^\pi \sin\vartheta_k d\vartheta_k \sigma_T(\vartheta_k,q^2,\vartheta,\phi)\Big/\left(\frac{\partial(q^2)}{\partial\phi_k}\right)\\
=&\frac{2k}{|\mathbf{p}_1-\mathbf{p}_2|}-\frac{k}{(\eta+\omega^2-q^2)D_1^{1/2}}\{-(\eta+\omega^2-q^2)^2+4(\eta+\omega^2-q^2)[\epsilon_1^2+\epsilon_2(\epsilon_2+\omega)]\\
&\qquad+8k^2-2(\eta+2)[2\epsilon_1(\epsilon_1-\omega)+2\epsilon_2(\epsilon_2+\omega)+q^2-\omega^2-4]\}\\
&+\frac{k}{(\eta+\omega^2-q^2)D_2^{1/2}}\{-(\eta+\omega^2-q^2)^2+4(\eta+\omega^2-q^2)[\epsilon_2^2+\epsilon_1(\epsilon_1-\omega)]\\
&\qquad+8k^2-2(\eta+2)[2\epsilon_1(\epsilon_1-\omega)+2\epsilon_2(\epsilon_2+\omega)+q^2-\omega^2-4]\}\\
&-\frac{k[4\epsilon_2(\epsilon_2+\omega)+q^2-\omega^2-4]}{D_1^{3/2}}[(q^2-q_2^2)[\eta+2\epsilon_1(\epsilon_1-\epsilon_2)]+4k\epsilon_1p_2^2\sin^2\vartheta]\\
&\qquad+\frac{k}{D_2^{3/2}}[4\epsilon_1(\epsilon_1-\omega)+q^2-\omega^2-4][(q^2-q_1^2)[\eta+2\epsilon_2(\epsilon_2-\epsilon_1)]-4k\epsilon_2p_1^2\sin^2\vartheta]-\frac{\omega^2}{q^2}I_L(q^2), \quad (22)
\end{aligned}$$

where

$$\begin{aligned}
&q_m=|\mathbf{p}_1-\mathbf{p}_2|-k,\quad q_M=|\mathbf{p}_1-\mathbf{p}_2|+k,\\
&\eta=(\mathbf{p}_1-\mathbf{p}_2)^2-(\epsilon_1-\epsilon_2)^2,\\
&q_1^2=\omega^2+\eta-2k[\epsilon_2-(\epsilon_1/p_1)p_2\cos\vartheta],\\
&q_2^2=\omega^2+\eta+2k[\epsilon_1-(\epsilon_2/p_2)p_1\cos\vartheta],\\
&D_1=p_1^2(q^2-q_2^2)^2+4k^2p_2^2\sin^2\vartheta,\\
&D_2=p_2^2(q^2-q_1^2)^2+4k^2p_1^2\sin^2\vartheta.
\end{aligned} \quad (23)$$

We note that the integrations performed in going from Eq. (18) to Eq. (20) have been made without approximation. In Eq. (20) we again verify that by setting $k=\epsilon_1-\epsilon_2$ ($\omega=0$) in our expressions (21) and (22) we obtain the quantities R_{ch} and R_{mag} given by Ginsberg and Pratt[2] for the elastic case:

$$I_L(\omega=0)=kR_{\mathrm{ch}},\quad I_T(\omega=0)=kR_{\mathrm{mag}}.$$

IV. HIGH-ENERGY LIMIT OF THE CROSS SECTION

We now consider the high-energy limit of Eq. (20), neglecting terms of relative order $1/\epsilon_2^2$. In taking this limit we place no restriction upon the order of magnitude of the excitation energy ω, it can have any value between 0 and ϵ_1. For $\epsilon_1 \gg 1$, $\epsilon_2 \gg 1$, and $\sin\vartheta$ of order unity ($\sin\vartheta \gg 1/\epsilon_2$) the expression (23) can be rewritten:

$$\begin{aligned} &\eta \approx 4\epsilon_1\epsilon_2 \sin^2\tfrac{1}{2}\vartheta\,,\\ &q_1^2 \approx \omega^2 + 4\epsilon_1(\epsilon_1-\omega)\sin^2\tfrac{1}{2}\vartheta\,,\\ &q_2^2 \approx \omega^2 + 4\epsilon_2(\epsilon_2+\omega)\sin^2\tfrac{1}{2}\vartheta\,,\\ &D_1 \approx \epsilon_1^2(q^2-q_2^2)^2 + 4k^2\epsilon_2^2\sin^2\vartheta\,,\\ &D_2 \approx \epsilon_2^2(q^2-q_1^2)^2 + 4k^2\epsilon_1^2\sin^2\vartheta\,. \end{aligned} \tag{24}$$

The entire discussion of Sec. III of I concerning the order of magnitude of the various terms applies here without change. It should be noted, however, that in addition to the high-energy approximations, $\epsilon_1 \gg 1$, $\epsilon_2 \gg 1$, stated explicitly there, the assumption $k \gg 1$ was also made throughout that discussion. However, as shown in Sec. V of this paper, no restriction on k is in fact required for the applicability of the separation into peak and background contributions, and the final high-energy cross section, (25), is valid (neglected terms being of relative order $1/\epsilon_2^2$) provided only that $\epsilon_1 \gg 1$, $\epsilon_2 \gg 1$, and that ϑ is not near 0°: $\vartheta \gg 1/\epsilon_2$. As in the elastic case, the cross section will be strongly peaked about the values q_1 and q_2 of the momentum transfer corresponding to photon emission in either the direction of the initial or final electron. The contribution to the cross section from these peaks in the integrand in Eq. (20) is denoted by P, and is of order $\ln\epsilon$. The contribution to the cross section from photons emitted in all other directions, the background contribution, is denoted by B. As in I [Eqs. (8) to (15) in that paper] we separate the contribution of the peaks from that of the background, writing explicit expressions for each of these contributions. The high-energy limit of the cross section given by (20) may then be written in the form

$$d\sigma = \frac{1}{2\pi}\frac{e^2}{\hbar c}\left(\frac{Ze^2}{mc^2}\right)^2 \frac{dk}{k}\frac{\sin\vartheta d\vartheta d\phi}{\epsilon_1^2}\{P+B\}\,, \tag{25}$$

where the contribution P from the peaks is

$$P = \frac{1}{2}\frac{\cos^2\frac{1}{2}\vartheta}{\sin^4\frac{1}{2}\vartheta}\left\{S(q_2)\left[\frac{\epsilon_1^2+(\epsilon_2+\omega)^2}{(\epsilon_2+\omega)^2}\ln 2\epsilon_1 - \frac{\epsilon_1}{\epsilon_2+\omega}\right] + S(q_1)\left[\frac{\epsilon_2^2+(\epsilon_1-\omega)^2}{(\epsilon_1-\omega)^2}\ln 2\epsilon_2 - \frac{\epsilon_2}{\epsilon_1-\omega}\right]\right\}, \tag{26}$$

with

$$S(q) = (1-\alpha)^2\mathfrak{F}_L^2(q) + (1-\alpha)\mathfrak{F}_T^2(q) + 2\mathfrak{F}_T^2(q)\tan^2\tfrac{1}{2}\vartheta \tag{27}$$

and

$$\alpha = \omega^2/q^2\,.$$

Here it seems interesting to mention that in the case of inelastic electron scattering without photon emission the high-energy limit of the cross section given by Alder *et al.*[3] may be written quite simply in terms of the expression $S(q)$ given above:

$$\frac{d\sigma}{d\Omega} = \left(\frac{Ze^2}{2\epsilon_1 mc^2}\right)^2 \frac{\cos^2\frac{1}{2}\vartheta}{\sin^4\frac{1}{2}\vartheta}S(q)\,. \tag{28}$$

If in $S(q)$ we suppose $\omega^2 \ll q^2$ we find the expression commonly used in the analysis of inelastic electron scattering experiments to separate the two form factors.[9]

The background contribution B is given by

$$\begin{aligned} B = {} & 4\epsilon_2^2[\epsilon_1^2+(\epsilon_2+\omega)^2]\cos^2\tfrac{1}{2}\vartheta \int_{q_m^2}^{q_M^2} d(q^2)\frac{[G(q)-G(q_2)]}{|q^2-q_2^2|}\\ & + 4\epsilon_1^2[\epsilon_2^2+(\epsilon_1-\omega)^2]\cos^2\tfrac{1}{2}\vartheta \int_{q_m^2}^{q_M^2} d(q^2)\frac{[G(q)-G(q_1)]}{|q^2-q_1^2|}\\ & + \int_{q_m^2}^{q_M^2} M(q)d(q^2) + \frac{[\epsilon_1(\epsilon_1-\omega)+\epsilon_2(\epsilon_2+\omega)]}{2\epsilon_1\epsilon_2}\\ & \times \frac{\cos^2\frac{1}{2}\vartheta}{\sin^4\frac{1}{2}\vartheta}S(\tilde{q})\ln\sin^2\tfrac{1}{2}\vartheta\,, \end{aligned} \tag{29}$$

with

$$\begin{aligned} M(q) = k\{ & (\epsilon_1-\epsilon_2)[1-2\eta(\eta+\omega^2-q^2)^{-1}]\\ & -2\epsilon_1\epsilon_2[\eta+(\epsilon_1-\epsilon_2)^2]^{-1/2}\}H(q)/q^4\\ & -8\epsilon_1\epsilon_2[\epsilon_1(\epsilon_1-\omega)+\epsilon_2(\epsilon_2+\omega)]\\ & \times\cos^2\tfrac{1}{2}\vartheta[G(q)-G(\tilde{q})]/(\eta+\omega^2-q^2) \end{aligned} \tag{30a}$$

for

$$q_m^2 < q^2 < q_2^2\,,$$

$$M(q) = k\{\epsilon_1+\epsilon_2-2\epsilon_1\epsilon_2[\eta+(\epsilon_1-\epsilon_2)^2]^{-1/2}\}H(q)/q^4 \tag{30b}$$

for

$$q_2^2 < q^2 < q_1^2\,,$$

$$\begin{aligned} M(q) = -k\{ & (\epsilon_1-\epsilon_2)[1-2\eta(\eta+\omega^2-q^2)^{-1}]\\ & +2\epsilon_1\epsilon_2[\eta+(\epsilon_1-\epsilon_2)^2]^{-1/2}\}H(q)/q^4\\ & +8\epsilon_1\epsilon_2[\epsilon_1(\epsilon_1-\omega)+\epsilon_2(\epsilon_2+\omega)]\\ & \times\cos^2\tfrac{1}{2}\vartheta[G(q)-G(\tilde{q})]/(\eta+\omega^2-q^2) \end{aligned} \tag{30c}$$

for

$$q_1^2 < q^2 < q_M^2\,,$$

where

$$\begin{aligned} & q_2^2 < \tilde{q}^2 = \omega^2 + 4\epsilon_1\epsilon_2\sin^2\tfrac{1}{2}\vartheta < q_1^2\,,\\ & G(q) = (q^2-\omega^2)^{-2}S(q)\,,\\ & H(q) = \mathfrak{F}_L^2(q) + (1-\alpha)^{-1}\mathfrak{F}_T^2(q) - 2(1-\alpha)^{-2}\mathfrak{F}_T^2(q)\,. \end{aligned} \tag{31}$$

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[9] See, for example, W. C. Barber, Ann. Rev. Nucl. Sci. **12**, 1 (1962).

We have written the integrals in B in such a way that the lack of singularity in the integrand at $q^2=\eta+\omega^2$ is clear.

As discussed in greater detail in Sec. IV of I, there are kinematic corrections of relative order $\epsilon m/M$ and dynamic corrections of relative order $(\epsilon/\ln\epsilon)(Zm/M)$ due to nuclear recoil. These corrections being smaller than those introduced by using the Born approximation, for the energy of experimental interest (30–1000 MeV), we do not include them in this calculation.

V. THE CASE OF 180° SCATTERING ANGLE; CONSIDERATION OF THE INFRARED DIVERGENCE

In our discussion of the high-energy approximation, and in particular in our estimates of the order of magnitude of the various terms in the cross section, we have in fact assumed not only that the electron energies are large: $\epsilon_1\gg1$, $\epsilon_2\gg1$, but in addition that the photon energy is large: $k\gg1$, and that the scattering angle is neither near 0° nor near 180°: $\sin\vartheta=O(1)$. It is desirable, however, that we have a high-energy expression for the cross section assuming only $\epsilon_1\gg1$, $\epsilon_2\gg1$, valid as well for scattering angles near or equal to 180° and for arbitrary photon energy. On one hand, electron scattering experiments in the extreme backward direction[10] require the validity for scattering angles near or equal to 180°. On the other hand, an analysis of the radiative tail in the neighborhood of either the elastic or one of the inelastic peaks, for ϵ_2 close to $\epsilon_1-\omega$ (and hence $k=\epsilon_1-\epsilon_2-\omega$ small), necessitates an expression for the cross section without restriction as to photon energy. This is also needed if one is to be able to integrate over the final energy of the electron in an energy region containing the elastic or one of the inelastic peaks, the addition of the radiative correction to the radiative tail giving then a finite integral. Indeed, if we integrate our cross section, (25), over the energy of the final electron to an upper limit $\epsilon_2=\epsilon_1-\omega-\Delta k$, then for small Δk we have

$$\begin{aligned}\int_{\epsilon_2}^{\epsilon_1-\omega-\Delta k}\frac{d\sigma}{dk}d\epsilon_2 &=\frac{1}{2\pi}\frac{e^2}{\hbar c}\left(\frac{Ze^2}{mc^2}\right)^2\frac{d\Omega}{\epsilon_1^2}\int_{\Delta k}^{\epsilon_1-\omega-\epsilon_2}\{P+B\}\frac{dk}{k}\\ &=\frac{1}{2\pi}\frac{e^2}{\hbar c}\left(\frac{Ze^2}{mc^2}\right)^2\frac{d\Omega}{\epsilon_1^2}\ln(1/\Delta k)\{P(k=0)+B(k=0)\}\\ &\quad+\text{terms which remain finite as }\Delta k\to0.\end{aligned}\tag{32}$$

The terms with the factor $\ln(1/\Delta k)$ all come from the peak contribution (26) and from the last term in the background contribution (29). In fact, in (29) we separated off this term in the integrand, integrating it in closed form, for just this purpose. The terms with the factor $\ln(1/\Delta k)$ in the cross section integrated over the energy of the final electron are thus simply

$$\frac{1}{2\pi}\frac{e^2}{\hbar c}\left(\frac{Ze^2}{mc^2}\right)^2\frac{d\Omega}{\epsilon_1^2}\frac{\cos^2(\frac{1}{2}\vartheta)}{\sin^4(\frac{1}{2}\vartheta)}S(q_0)\times\ln(1/\Delta k)[\ln(q_0^2-\omega^2)-1],\tag{33}$$

where $q_0^2=\omega^2+4\epsilon_1(\epsilon_1-\omega)\sin^2(\frac{1}{2}\vartheta)$ is the momentum transfer to the nucleus in the limit $k=0$. Expression (33) with opposite sign is precisely the $\ln(1/\Delta k)$ term in the radiative correction to the cross section for electron scattering, which has been given for the case of elastic scattering from a Coulomb potential by Schwinger[11] and for inelastic scattering with both Coulomb and current interactions by Meister and Griffy.[12] All the logarithmically divergent terms thus cancel upon addition of the radiative correction, which accounts for the emission and reabsorption of virtual photons and the emission of real, soft, photons of energy less than Δk, to the integral of the radiative tail, which accounts for the emission of real, hard, photons of energy greater than Δk. We see, therefore, that by including *both* the peak and background contributions we do not have to introduce the *ad hoc* "calibration" suggested by Schiff.[13]

We could, of course, use the exact Born approximation cross section, Eq. (20), in which no approximations have been made. This expression is, however, far more complicated than is needed, given the high-energy conditions $\epsilon_1\gg1$, $\epsilon_2\gg1$ which are satisfied in the experiments of interest. Solely with these latter restrictions, the separation of the cross section into peak and background contributions remains, as will be shown, extremely useful, giving clearly the order of magnitude of each of the terms and allowing us to neglect all contributions which are of relative order $1/\epsilon_2^2$. Furthermore, the background integrals, which must finally be computed numerically for an arbitrary form factor, are, by this procedure, put in a form such that the accuracy of standard integration techniques is adequate. In fact we shall find, after an analysis of which the essential steps will be given, that the cross section as given in Eq. (25) is valid for arbitrary k and for ϑ close or equal to 180°, the only restrictive conditions being $\epsilon_1\gg1$, $\epsilon_2\gg1$, and $\vartheta\gg1/\epsilon_2$.

We return to the cross section (20) and examine again the various terms appearing there, but now proceeding further before making any approximations. We consider first the terms with factor $D_{1,2}^{-1/2}\mathfrak{F}_L^2$ or

[10] G. A. Peterson and W. C. Barber, Phys. Rev. **128**, 812 (1962); J. Goldemberg and W. C. Barber, *ibid.* **134**, B963 (1964).

[11] J. Schwinger, Phys. Rev. **75**, 898 (1949); J. W. Motz, Haakon Olsen, and H. W. Koch, Rev. Mod. Phys. **36**, 881 (1964).

[12] N. T. Meister and T. A. Griffy, Phys. Rev. **133**, B1032 (1964).

[13] L. I. Schiff, Phys. Rev. **87**, 750 (1952).

$D_{1,2}^{-1/2}\mathfrak{F}_T^2$, which are, as previously noted in I, of the form

$$\int_{x_m}^{x_M}\frac{f(x)dx}{[(x-x_0)^2+a^2]^{1/2}}=f(x_0)\int_{x_m}^{x_M}\frac{dx}{[(x-x_0)^2+a^2]^{1/2}}+\int_{x_m}^{x_M}\frac{[f(x)-f(x_0)]dx}{[(x-x_0)^2+a^2]^{1/2}}, \quad (34)$$

where, as before, we denote the first integral on the right-hand side as the peak contribution P_0, and the second as the background contribution B_0. Evaluating the first integral without any approximation we find

$$\int_{x_m}^{x_M}\frac{dx}{[(x-x_0)^2+a^2]^{1/2}}=2\ln(\epsilon_1+p_1) \quad (35)$$

for the terms with factor $D_1^{-1/2}$. [Note, from (21) and (22), that the terms with factor $D_2^{-1/2}$ or $D_2^{-3/2}$ may be obtained from the corresponding terms with factor $D_1^{-1/2}$ or $D_1^{-3/2}$ by the substitutions $\epsilon_1\rightleftarrows\epsilon_2$, $k\to -k$, $\omega\to-\omega$.] We then note that the order of magnitude of the background contribution from terms with factor $\mathfrak{F}_T^2$ is independent of k and ϑ (provided only that $\vartheta\gg 1/\epsilon_2$) and that the order of magnitude of the background contribution from terms with factor $\mathfrak{F}_L^2$ is diminished only if both $k/\epsilon_2\ll 1$ and $\pi-\vartheta\lesssim 1/\epsilon_2$. Neglecting terms in the background contribution which are of order $1/\epsilon_2^2$ relative to their value for $k/\epsilon_2=O(1)$ and $\sin\vartheta=O(1)$ then gives Eq. (29). Although these order of magnitude considerations must be made before neglect of any terms, their plausibility is indicated in a simple fashion by Eq. (29): Noting, from (23), that $q_M^2-q_m^2=O(k\epsilon)$, we see that the first two integrals in (29), with factor $\cos^2(\frac{1}{2}\vartheta)$, are of order $(k/\epsilon)\cos^2(\frac{1}{2}\vartheta)$ and that the third integral is of order k/ϵ. The last term in (29) is of order $\cos^2(\frac{1}{2}\vartheta)\mathfrak{F}_L^2+\mathfrak{F}_T^2$ for all k.

We next note that the order of magnitude of the peak contribution is independent of k, and that the peak contribution from terms with factor $\mathfrak{F}_T^2$ is of order $\ln\epsilon$ relative to the background integral for all $\vartheta\gg 1/\epsilon_2$. The peak contribution from terms with factor $\mathfrak{F}_L^2$ is, however, of order $\ln\epsilon$ relative to the background integral only for $\sin\vartheta=O(1)$, but of relative order $\epsilon_2^{-2}\ln\epsilon_2$ for $\pi-\vartheta\lesssim O(1/\epsilon_2)$. (We note that these peak terms do not in fact vanish, but remain of relative order $\epsilon_2^{-2}\ln\epsilon_2$, for $\vartheta=\pi$.) We then neglect, in the peak contribution, all terms which for all k and $\vartheta\gg 1/\epsilon_2$ are of order ϵ_2^{-2} or $\epsilon_2^{-2}\ln\epsilon_2$ relative to the value of the peak contribution for $\sin\vartheta=O(1)$.

We next proceed to the terms with factor $D_{1,2}^{-3/2}\mathfrak{F}_L^2$ or $D_{1,2}^{-3/2}\mathfrak{F}_T^2$. However, we now write these terms in a form slightly different from that in I, namely

$$\int_{x_m}^{x_M}\frac{b(x)f(x)dx}{[(x-x_0)^2+a^2]^{3/2}}=b(x_0)f(x_0)\int_{x_m}^{x_M}\frac{dx}{[(x-x_0)^2+a^2]^{3/2}}+f(x_0)\int_{x_m}^{x_M}\frac{[b(x)-b(x_0)]dx}{[(x-x_0)^2+a^2]^{3/2}}+\int_{x_m}^{x_M}\frac{b(x)[f(x)-f(x_0)]dx}{[(x-x_0)^2+a^2]^{3/2}}. \quad (36)$$

Here we choose

$$b(x)=2kp_2^2\epsilon_1^2\sin^2\vartheta+p_1(p_1-p_2\cos\vartheta)(x-x_0)$$

for terms with factor $D_1^{-3/2}$, noting that $b(x)$ occurs as a factor in all these terms and that $b(x_0)$ has the factor $\sin^2\vartheta$. We again call the first terms on the right-hand side of (36) the peak contribution, P_1, and now denote by first, B_1, and second, B_2, background contribution, respectively, the two remaining integrals. The second background integral B_2 may be shown to be of order $\epsilon_2^{-2}\ln\epsilon_2$ relative to B_0 for all k and $\vartheta\gg 1/\epsilon_2$, and hence may be neglected. The integrals appearing in the peak P_1 and first background B_1 contribution may be evaluated without approximation and are, for terms with factor $D_1^{-3/2}$,

$$b(x_0)f(x_0)\int_{x_m}^{x_M}\frac{dx}{[(x-x_0)^2+a^2]^{3/2}}=-\frac{2f(x_0)p_2^2\epsilon_1^2\sin^2\vartheta}{p_1^2-2p_1p_2\cos\vartheta+p_2^2+p_1^2p_2^2\sin^2\vartheta}\equiv P_1, \quad (37)$$

$$f(x_0)\int_{x_m}^{x_M}\frac{[b(x)-b(x_0)]dx}{[(x-x_0)^2+a^2]^{3/2}}=-\frac{2f(x_0)(p_1^2-2p_1p_2\cos\vartheta+p_2^2-p_2^2\sin^2\vartheta)}{p_1^2-2p_1p_2\cos\vartheta+p_2^2+p_1^2p_2^2\sin^2\vartheta}\equiv B_1. \quad (38)$$

For $\sin\vartheta=O(1)$, the first background contribution B_1 is thus of order $1/\epsilon_2^2$ relative to the peak contribution and may be neglected. Further, if $\pi-\vartheta\lesssim O(1/\epsilon_2)$, then for the terms with factor $\mathfrak{F}_L^2$, both P_1 and B_1 are of order $1/\epsilon_2^2$ relative to B_0 because of the term $4\epsilon_2(\epsilon_2+\omega)+\omega^2-q_2^2$ or $4\epsilon_1(\epsilon_1-\omega)+\omega^2-q_1^2$ in $f(x_0)$. This is, however, not the case for terms with factor $\mathfrak{F}_T^2$. For these terms, if $\pi-\vartheta=O(1/\epsilon_2)$ then P_1, B_1, and B_0 are all of the same order and hence must be kept, and if $\vartheta=\pi$ then $P_1=0$, and B_1 and B_0 are of the same order. Thus, in order to have a cross section valid also for $\pi-\vartheta\lesssim O(1/\epsilon_2)$ we must keep both P_1 and B_1. The sum of these two contributions is a very simple expression: $P_1+B_1=-2f(x_0)$, which is just the value of P_1 for

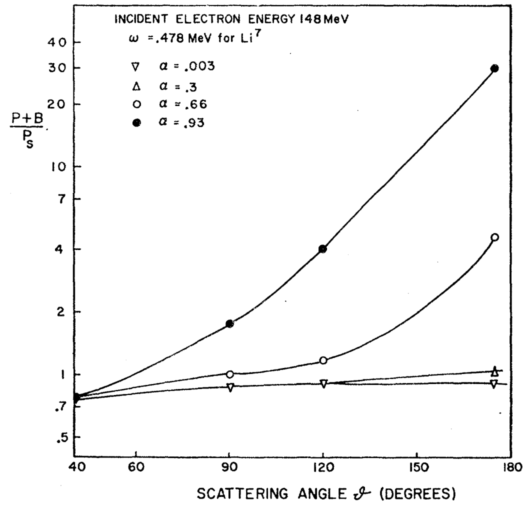

FIG. 1. The ratio $(P+B)/P_S$ as a function of the scattering angle for different values of $\alpha \equiv k/\epsilon_1$ and for 148-MeV incident electron energy, P_S being the sum of the logarithmic terms in P.

$\epsilon_1 \gg 1$, $\epsilon_2 \gg 1$, and $\sin\vartheta = O(1)$. For this reason, as well as those given in the discussion of this section, the expression (25) for the cross section, previously derived assuming $k \gg 1$ and $\sin\vartheta = O(1)$, is in fact valid for arbitrary k and $\vartheta \gg 1/\epsilon_2$. We see, however, that the terms B_1, which are negligible for $\sin\vartheta = O(1)$, must be retained for $\pi - \vartheta \lesssim O(1/\epsilon_2)$.

For the numerical calculations to follow, the advantage of the above procedure is that we have integrated in closed form the highly peaked terms with factor $D_1^{-3/2}$ or $D_2^{-3/2}$, and put the integral B_0 in a form such that there are no peaks in the integrand.

VI. NUMERICAL EVALUATION AND CONCLUSIONS

To show the relative importance of the peak and background contributions we have performed the calculation of the radiative tail for a particular case. We chose the 0.478-MeV level of Li^7 as this case has been studied experimentally by Bernheim and Bishop.[14] Those authors verified that the experimental values of the form factors for this level are in very good agreement with the theoretical values calculated by Willey[15] using an odd-proton model. Following Willey we can write the two form factors for this particular level as

$$\mathfrak{F}_L{}^2 = (0.44Z^{-2})x^2 e^{-2x}[1-0.572(\hbar cq/Mc^2)^2]^2, \quad (39)$$

$$\mathfrak{F}_T{}^2 = (0.81Z^{-2})(\hbar cq/Mc^2)^2 x^2 e^{-2x}, \quad (40)$$

where $x = q^2/4\beta$ with q^2 expressed in F^{-2} and β^{-1}, which is the oscillator radial scale parameter, taken equal to 4 F^2 as suggested by the elastic scattering data.

In Fig. 1 we have plotted the ratio of the complete expression $P+B$ [Eqs. (26) and (29)] to the sum of the logarithmic terms in P, which are the only ones to remain in the Schiff approximation.[13] However, as long as one considers the emission of photons of large energy, the absolute cross section of the radiative tail is very small. The procedure generally followed in this type of calculation (taking into account only the logarithmic terms in the peak contribution in the integrated cross section) is then not too crucial. This can be seen in Fig. 2 where we have plotted the cross section as given by Eq. (25) for the radiative tail of the inelastic level as a function of the final energy for fixed initial energy and fixed scattering angle, using the form factors given in Eqs. (39) and (40). On the same figure we have also plotted the equivalent cross section for the elastic scattering, using Eq. (25) in which we set $k = \epsilon_1 - \epsilon_2$ $(\omega = 0)$. For this case we used, in the numerical calculations, the following expression for the longitudinal form factor:

$$\mathfrak{F}_L = (1 - 0.895 \times 10^{-6} q^2) \exp(-0.464 \times 10^{-5} q^2), \quad (41)$$

which is given by the independent-particle shell model[16] assuming a radius of 2.1 F for Li^7. As we had no theoretical expression available for the transverse form factor, we arbitrarily chose

$$\mathfrak{F}_T = 0.1\mathfrak{F}_L. \quad (42)$$

This choice does not influence very much the final values as the calculation has been performed for a scattering angle far from 180°.

Throughout this paper we have been considering the case of an electron inelastically scattered by a nucleus left in an excited state. We should like to point out that as long as the excited state has a fixed multi-

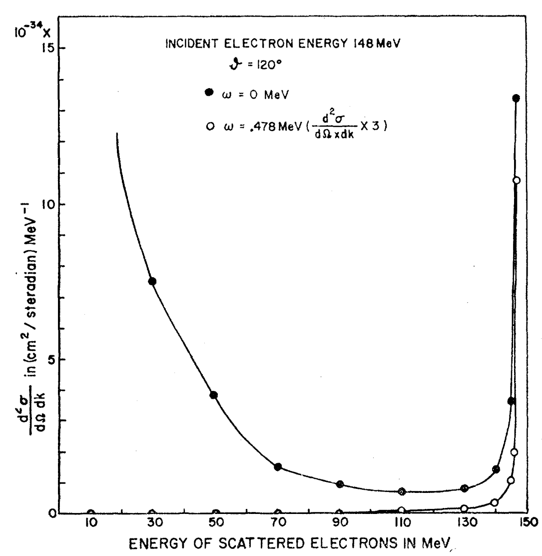

FIG. 2. The cross section calculated from Eq. (25) both for the elastic and first inelastic level for incident electron energy 148 MeV and scattering angle 120°.

[14] M. Bernheim and G. R. Bishop, Phys. Letters **5**, 294 (1963).
[15] R. S. Willey, Nucl. Phys. **40**, 529 (1963).
[16] R. Hofstadter, Ann. Rev. Nucl. Sci. **7**, 231 (1957).

polarity our calculation is applicable independently of the way in which the nucleus becomes deexcited. In particular, our calculation can be applied to electrodisintegration processes such as that occuring in the giant resonance transition. This will also apply to the case of quasielastic scattering or pion electroproduction where only the scattered electrons are detected, as in that case there should be no interference between different multipoles.[17] However, if any of the products of electrodisintegration is detected, then the problem must be studied in greater detail.

[17] W. C. Barber, F. Berthold, G. Fricke, and F. E. Gudden, Phys. Rev. **120**, 2081 (1960). See Ref. 21 to G. Kramer in this reference.

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Nuclear Structure Studies in Tellurium Isotopes with (d,p) and (d,t) Reactions*

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Shell-model states in the isotopes 123, 124, 125, 126, 127, 129, and 131 of Te have been investigated via (d,p) and (d,t) reactions with a typical resolution of 40 keV. Distorted-wave Born approximation calculations were used for the identification of the orbital angular momentum of the captured neutron in (d,p) reactions. The $11/2^-$, $3/2^+$ and $1/2^+$ neutron subshells in the 50–82 neutron shell are found to be filling in these isotopes, although the $3/2^+$ and $1/2^+$ subshells are not filling as rapidly as one would normally expect. There is also found some indication of the $7/2^-$ and the $3/2^-$ subshells from the next major shell filling in the heaviest isotopes. The $3/2^+$ single-particle state is found to lie the lowest in the isotopes 131 and 129, while $11/2^-$ and $1/2^+$ are found to lie the lowest in the isotopes 127 and 125, respectively. Single-particle energies have been calculated using pairing theory.

I. INTRODUCTORY—EXPERIMENTAL PROCEDURE

According to the shell-model theory[1] the neutron single-particle states are practically unaffected by the proton number as long as it is even. One, therefore, hopes to see about similar spectra of neutron states in neighboring isotones. Previous work of Cohen and Price[2] and some present work of Schneid, Prakash, and Cohen[3] with Sn isotopes has revealed a simple level structure of their neutron states which makes these isotopes very suitable for shell model studies via (d,p) and (d,t) reactions. From the aforesaid remark about proton number, Te isotopes (which have only two more protons than the Sn isotopes) should show a structure similar to the Sn isotopes and are, therefore, very suitable for shell-model studies in the mass region $A \sim 125$.

Te isotopes 124, 125, 126, 128, and 130 were deposited by vacuum evaporation on gold foils (~ 0.2 mg/cm^2) to thicknesses varying from 0.5 to 1 mg/cm^2. The thickness of these targets may be uncertain by $\sim 10\%$ at the most. These isotopes were bombarded with 14.8-MeV deuterons from the University of Pittsburgh 47-in. fixed-frequency cyclotron. The reaction products were analyzed by a 60° wedge magnet spectrograph and detected in nuclear emulsion plates. Other details of the experimental method have been described previously.[4]

At the time this experiment was done, it was believed from the previous work of Cohen and Price[2] that measurement of a complete angular distribution in (d,p) reactions is no more helpful than cross-section measurements at a few key angles judiciously chosen to give information about l_n, the orbital angular momentum of the captured neutron. In the same work it was realized that (d,t) angular distributions are much less useful, as these were found to be relatively insensitive to differences in angular momenta as compared to (d,p) angular distributions. Hence data were taken at 9, 17, 20, 30, 39, and 50 deg for (d,p) reactions and only at 45 and 60 deg for (d,t) reactions. The former choice was made by an examination of the (d,p) angular distributions for the Sn isotopes from the work of Cohen and Price[2] while the

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